Convex Choice

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Introduction (1)

Agent with utility $u(a, \theta)$, $a \in A$ and $\theta \in \Theta \subset \mathbb{R}^n$

Important result in 1-dim signaling & mech design

$\rightarrow$ IC reduces to local IC under single-crossing property ($\equiv$ "interval choice")

How to extend to multi-dim types?

This paper: convex choice

$\rightarrow$ from any choice set, any action is chosen by a convex set of types

Natural requirement; useful even beyond local IC

Introduction (2)

Main results:

1. Sense in which convex choice characterizes sufficiency of local IC
2. Other applications: implementability; cheap talk
3. Convex choice $\iff$ "directional single crossing"
4. For EU on lotteries, convex choice $\implies$ "one-dim or affine" representation
$$u(a, \theta) = v(a) \cdot \theta + w(a)$$

This affine form has been salient in multi-dim studies

Related Literature

Convex choice: Grandmont 1978

Interval choice and lotteries: Kartik, Lee, Rappoport 2024

Multi-dim single crossing: McAfee & McMillan 1988; Milgrom & Shannon 1994

Applications

- Sufficiency of local IC: Carroll 2012
- Implementability: Saks & Yu 2005; BCLMNS 2006
- Cheap talk: Levy & Razin 2004; Sobel 2016

Convex Choice

and

Applications

Convex Choice

Agent with utility $u(a, \theta)$, $a \in A$ and $\theta \in \Theta \subset \mathbb{R}^n$, Θ convex

Definition

u has convex choice if $\forall B \subset A$ and $\forall a \in B$,

$$\left\{\theta : \{a\} = \operatorname*{argmax}_{b \in B} u(b, \theta)\right\} \text{ is convex.}$$

(Enough to only consider all binary choice sets)

- Grandmont's 1978 "betweeness"
- In 1-dim, "interval choice" of Kartik, Lee, Rappoport 2024

Convex Choice

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For talk, maintain "regular" indifferences:

$$\left[u(a', \theta') > u(a'', \theta') \text{ and } u(a', \theta'') = u(a'', \theta'')\right] \implies u(a', \theta) > u(a'', \theta) \quad \forall \theta \in (\theta', \theta'').$$

Satisfied, e.g., by no indifferences or by $A \subset \mathbb{R}^n$ and $u(a, \theta) = a \cdot \theta$

(Paper uses a weaker version, and selectively.)

Incentive Compatibility

$N_\theta \subset \Theta$ denotes open neighborhood of θ (in relative topology)

Direct mechanisms $\Theta \to A$ (subsumes stochastic mechs)

Definition

Mechanism $m : \Theta \to A$ is

- incentive compatible (IC) if $\forall \theta \in \Theta$,

$$\forall \theta' \in \Theta : u(m(\theta), \theta) \geq u(m(\theta'), \theta) \text{ and } u(m(\theta'), \theta') \geq u(m(\theta), \theta').$$

- locally IC if $\forall \theta \in \Theta$, $\exists N_\theta \subset \Theta$ s.t.

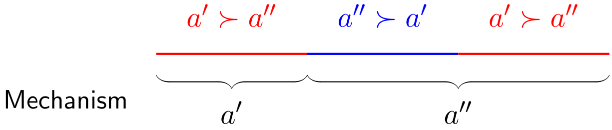
$$\forall \theta' \in N_\theta : \; u(m(\theta), \theta) \geq u(m(\theta'), \theta) \text{ and } u(m(\theta'), \theta') \geq u(m(\theta), \theta').$$

Analogously for a mechanism defined on $\Theta' \subset \Theta$

(Elaborate)

Sufficiency of Local IC (1)

Local IC does not generally imply IC:



Not convex choice!

Sufficiency of Local IC (2)

Proposition

u has convex choice $\implies$ if $m : \Theta \to A$ is locally IC then it is IC.

$\Updownarrow$

for any line segment $\ell \subset \Theta$ and any mechanism $m : \ell \to A$, if m is locally IC then it is IC.

- So IC between θ and θ' requires only checking local IC along line segment (θ, θ')
- Such "integration up" is a common strategy
- Corollary: sufficiency of local IC on full type space Θ

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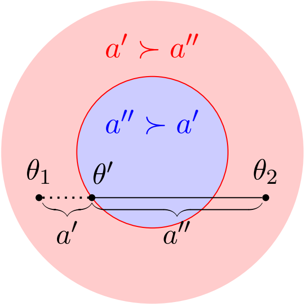
- Sufficiency of local IC on Θ $\;\not\Longrightarrow\;$ CC
- But sufficiency on all line segments does
- All line segments of interest because:
  1. Reduces checking IC between any two types to a 1-d task, which is 'tractable'
  2. A tractable problem must remain tractable when restricted to lower dimensions



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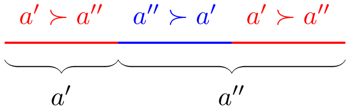
Proof idea: **Necessity** of CC captured by earlier 1-dim example



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Proof idea: Heuristic for **sufficiency**

Assume no indiff, take any θ, θ' and a fine grid on their line segment, $\theta = \theta_1, \ldots, \theta_n = \theta'$

- local IC $\implies$ $u(m(\theta_i), \theta_i) > u(m(\theta_{i+1}), \theta_i) \quad \forall i = 1, 2$
  $u(m(\theta_3), \theta_3) > u(m(\theta_2), \theta_3)$
- convex choice $\implies$ $u(m(\theta_1), \theta_1) > u(m(\theta_3), \theta_1)$
- iterate logic, using local IC and CC each time, to get $u(m(\theta_1), \theta_1) > u(m(\theta_1), \theta_n)$.

Sufficiency of Local IC (2)

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Carroll 2012 establishes sufficiency of local IC using "domain representation" of prefs

Our parameter representation approach is complementary

Formally, his result is subsumed by $A \subset \mathbb{R}^n$ and $u(a, \theta) = a \cdot \theta$

▸ Implementability

Cheap Talk

In cheap talk or costly signaling,

sender's utility having convex choice $\implies$ every eqm is "convex partitional"

(modulo details about indifferences)

Has been interest in extending Crawford & Sobel 1982 to multiple dims

- Levy & Razin 2004, 2007; Chakraborty & Harbaugh 2007

Also commmon-interest cheap talk with finite messsage space

- Jager, Metzger, Riedel 2011; Saint-Paul 2017; Sobel 2016; Bauch 2024

Remark

Assume $A \subset \mathbb{R}^n$ and $u(a, \theta) = -l(\|a - \theta\|)$, with $l(\cdot)$ strictly $\uparrow$.

(and $A \cap \Theta$ has nonempty interior)

Convex choice $\iff$ norm is weighted Euclidean

(i.e., $\|x\| = \sqrt{xWx^T}$, with W sym pos def)

Directional Single Crossing

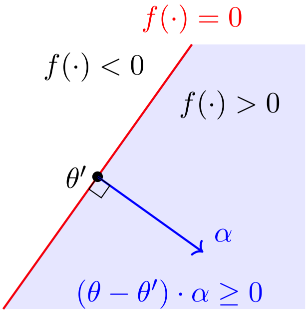
Directional Single Crossing (1)

Convex choice can be viewed as single crossing

Definition

$f : \Theta \to \mathbb{R}$ is directionally single crossing if $\exists \alpha \in \mathbb{R}^n \setminus \{0\}$ s.t. $\forall \theta, \theta' \in \Theta$,

$$(\theta - \theta') \cdot \alpha \geq 0 \implies \text{sign}\left(f(\theta)\right) \geq \text{sign}\left(f(\theta')\right).$$



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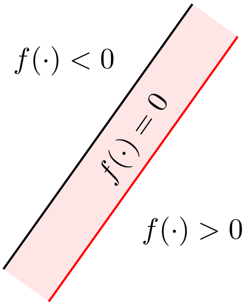

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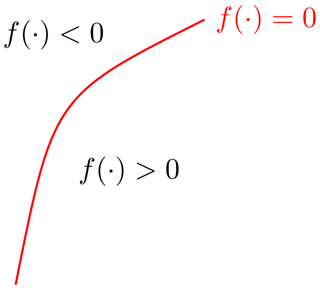

Not DSC

Directional Single Crossing (2)

Convex choice can be viewed as single crossing

Definition

$u : A \times \Theta \to \mathbb{R}$ has directionally single-crossing differences if $\forall a, a' \in A$,

$$u(a, \theta) - u(a', \theta) \text{ is directionally single crossing.}$$

- $\forall a, a'$, strict preference sets are parallel half-spaces, either open or closed (intersected with the type space)
- Direction of defining hyperplanes can vary across action pairs

Directional Single Crossing (2)

Convex choice can be viewed as single crossing

Definition

$u : A \times \Theta \to \mathbb{R}$ has directionally single-crossing differences if $\forall a, a' \in A$,

$$u(a,\theta) - u(a',\theta) \text{ is directionally single crossing.}$$

Leading example families, when $A \subset \mathbb{R}^n$:

1. weighted Euclidean: any $\downarrow$ fn of $(a-\theta)W(a-\theta)^T$, with W sym pos def
2. CES: $A, \Theta \subset \mathbb{R}^n_+$ and $u(a,\theta) = \left(\sum_{i=1}^n (a_i)^r \theta_i\right)^s$ with $r \in \mathbb{R}$ and $s > 0$

For these families, adding a type-independent function preserves DSCD, so, e.g.,

$u(a,\theta) = a \cdot \theta + w(a)$ has DSCD

Directional Single Crossing (3)

Convex choice can be viewed as single crossing

Proposition

If u has DSCD, then u has convex choice.

If u "strictly violates" DSCD, then u does not have convex choice.

- 1st statement straightforward from geometry
- 2nd follows from a sep hyp thm

- Closely related to Grandmont 1978; his form is more restrictive (e.g., continuity)

Convex Environments

Convex Environments (1)

Choice among lotteries with EU: $A \equiv \Delta X$ and $u(a, \theta) \equiv \sum_x a(x)\bar{u}(x, \theta)$

- stochastic or multiple-agent mechanisms
- cheap talk where sender is uncertain about receiver prefs

More generally, convex environment: $\{u(a, \cdot) : \Theta \to \mathbb{R}\}_{a \in A}$ is convex

- rank-dependent EU / prob distortion, where distortion function has convex image
- choice over T-period consumption streams:

  $A \equiv [\underline{a}, \overline{a}]^T$ and $u(a, \theta) \equiv \sum_t v(a_t)\rho(t; \theta)$, with $v(\cdot)$ continuous

Convex Environments (2)

Proposition

Assume $\Theta = \mathbb{R}^n$, $u(a, \theta)$ is differentiable in θ, and no type is totally indifferent.

Convex environment and DSCD $\implies$ u is 1-dimensional or has affine representation.

- 1-dimensional if $\exists \alpha \in \mathbb{R}^n \setminus \{0\}$ and $\tilde{u} : A \times \mathbb{R} \to \mathbb{R}$ s.t.
  $\tilde{u}(a, \alpha \cdot \theta)$ represents the same prefs for every θ
- Affine representation if $\exists v : A \to \mathbb{R}^n$ and $w : A \to \mathbb{R}$ s.t.
  $v(a) \cdot \theta + w(a)$ represents the same prefs for every θ

Convex Environments (2)

Proposition

Assume $\Theta = \mathbb{R}^n$, $u(a, \theta)$ is differentiable in θ, and no type is totally indifferent.

Convex environment and DSCD $\implies$ u is 1-dimensional or has affine representation.

- Consider CES prefs: $X, \Theta \subset \mathbb{R}^n_+$ (with nonempty interiors) and
  $\bar{u}(x, \theta) = \left(\sum_{i=1}^n (x_i)^r \theta_i \right)^s + w(x)$ with $r \in \mathbb{R}$ and $s > 0$.
- Although $\bar{u}$ satisfies DSCD, does the induced EU over $A = \Delta X$?
- If $n = 1$, yes. But when $n > 1$, if and only if $s = 1$.

Convex Environments (2)

Proposition

Assume $\Theta = \mathbb{R}^n$, $u(a, \theta)$ is differentiable in θ, and no type is totally indifferent.

Convex environment and DSCD $\implies$ u is 1-dimensional or has affine representation.

Conclusion also holds under alternate assumptions

- Prop 5: quasi-linear, differentiable in type, and minimally rich (drop $\Theta = \mathbb{R}^n$)

Interpretation:

- In rich environments, genuinely multi-dim prefs are unwieldy unless affine
- New perspective on why multi-dim mech design has emphasized affine form

Our exercise only allows changing representation; not redefining types

Conclusion

Convex choice is a valuable property

- characterizes sufficiency of local IC (on all line segments)
- other applications: implementability; cheap talk
- essentially equiv to a form of single crossing with simple geometric interpretation
- in convex envs with some regularity, "one-dimensional or affine representation"

(Others: preference aggregation; social learning)

Another interesting notion: connected choice

- also relevant for sufficiency of local IC (on full type space)
- we view convex choice as more appealing

Thank you!

On Local IC Definition

(Back)

Definition

Mechanism $m : \Theta \to A$ is locally IC if $\forall \theta \in \Theta$, $\exists N_\theta \subset \Theta$ s.t.

$$\forall \theta' \in N_\theta : \; u(m(\theta), \theta) \geq u(m(\theta'), \theta) \; \text{ and } \; u(m(\theta'), \theta') \geq u(m(\theta), \theta').$$

Example

$$a'' \succ a'$$

Mechanism [------ a' ------)[----- a'' ------]

Our defn is weaker than: $\exists \varepsilon > 0$ s.t. $\forall \theta \in \Theta$, $\exists B_\theta^\varepsilon$ s.t.

$$\forall \theta' \in B_\theta^\varepsilon \cap \Theta : \; u(m(\theta), \theta) \geq u(m(\theta'), \theta).$$

Implementability

$A \equiv Y \times \mathbb{R}$; assume Y finite. Quasilinear prefs: $u((y,t),\theta) \equiv \tilde{u}(y,\theta) - t$

(Back)

Allocation rule $\upsilon : \Theta \to Y$ is implementable if $\exists\ \tau : \Theta \to \mathbb{R}$ s.t. (υ, τ) is IC

Which allocation rules are implementable?

Necessary condition is weak (or 2-cycle) monotonicity:

$$\forall \theta, \theta' :\ \tilde{u}(\upsilon(\theta),\theta) - \tilde{u}(\upsilon(\theta'),\theta) \geq \tilde{u}(\upsilon(\theta),\theta') - \tilde{u}(\upsilon(\theta'),\theta')$$

(Rochet 1987: "cyclical monotonicity" is nec & suff)

Saks & Yu 2005: weak mon is suff if Θ convex, $Y \subset \mathbb{R}^n$, and $\tilde{u}(y,\theta) = y \cdot \theta$

Proposition

Assume u has convex choice and is continuous in θ.

Every weakly monotone allocation rule is implementable.

Proof uses result from

Berger, Müeller, Naeemi 2017