A Result on Optimal Deterministic Allocation Rules*

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There is a principal and an agent. The principal has to make a binary decision: whether (or not) to allocate an object to the agent, which we also refer to as accepting the agent. The agent has a private type $\theta \in \Theta$, drawn from some prior distribution, and chooses an observable action $a \in A(\theta) \subseteq \mathcal{A}$. The principal commits to a mechanism $\alpha : \mathcal{A} \to [0, 1]$, where $\alpha(a)$ is the acceptance probability when the agent chooses action a.

The agent's has some expected utility function that depends only on his type and whether he is accepted. So each type of the agent seeks to either minimize or maximize (or is indifferent over) the acceptance probability, but this preference need not be the same across types. Note that as in disclosure games, actions in $A(\theta)$ are costless while those not in $A(\theta)$ are unavailable.

Let G denote a joint distribution of actions a and types θ induced by the agent's strategy and the type distribution. The principal's goal is to maximize

$$\mathbb{E}[\alpha(a) \cdot \pi(\theta; a, G)] + V(G),$$

where $\pi(\theta; a, G)$ is the payoff of accepting an agent of type θ who takes action a and the joint distribution is G, and $V(G)$ is a direct payoff that depends only on the joint distribution.[1] The V component may represent direct type-dependent action payoffs

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[1] Throughout, we assume the relevant expectations / (Lebesgue) integrals exist.

to the principal, whereas π captures her payoff from acceptance, which can depends on the agent's type, his action, and the type-action distribution. The dependence of the acceptance payoff on the type-action distribution G can capture the judgment of a third party on the principal's decision that depends on G, as in Dessein, Frankel, and Kartik (2023).

Proposition 1. There is an optimal deterministic mechanism, i.e., there is $\alpha : \mathcal{A} \to \{0, 1\}$ that maximizes the principal's objective.[2]

Here is the intuition. Conditional on the joint distribution G of actions and types, the principal's payoffs are linear in the allocation, and so the principal ex post (weakly) prefers deterministic allocations. The only potential benefit of interior probabilities is to maintain the agent's incentives to take actions. But the agent's incentives depend only on the ordering of the allocation probabilities at different actions, not the cardinal values. So the principal can maintain action incentives with any monotone transformation of allocation probabilities. In particular, the principal can spread all of the probabilities to 0 and 1 while (weakly) maintaining their ordering: probabilities below a threshold map to 0, and above a threshold map to 1.

The formal proof shows that given any (possibly suboptimal) allocation rule, we can find an order-preserving spreading of probabilities to 0 and 1 that (i) maintains all action incentives, meaning it keeps the joint distribution G of actions and types unaffected, and (ii) weakly increases the principal's allocation payoff.

Proof. Fix any mechanism α and any incentive compatible (IC) agent strategy σ with type-action distribution G. Abusing notation, let $A(p) := \alpha^{-1}(p)$ be the set of actions that lead to acceptance probability p. For any p such that $A(p) \neq \emptyset$, let

$$u(p) := \mathbb{E}_G[\pi(\theta; a, G) | a \in A(p)]$$

be the principal's payoff if she were to instead accept all the actions that α assigns

[2] We assume that if the agent is indifferent among actions, he breaks ties in favor of the principal. This is innocuous, given that there is only one agent.

probability p to (holding the agent's strategy fixed). For p such that $A(p) = \emptyset$, the choice of $u(p)$ does not matter; for concreteness, we can set $u(p) := 0$. Let F be the cumulative distribution over acceptance probabilities induced by G and α. Let

$$U(p') := \int_0^{p'} u(p)\mathrm{d}F(p)$$

be the principal's payoff if she were to instead accept all the actions that α assigns probability less than p'. Note that because F is right-continuous, there are two (exclusive and exhaustive) cases: either (1) there is p^* such that $U(p^*) = \inf_p U(p)$; or (2) $\inf U(p)$ is not achieved and there is p^* such that $\lim_{p \nearrow p^*} U(p) = \inf_p U(p)$, i.e., U jumps up at p^*.

Case (1): Consider a modification of α to $\bar{\alpha}$ such that any action a with $\alpha(a) \geq p^*$ is assigned $\bar{\alpha}(a) = 1$, and for all other actions $\bar{\alpha}(a) = \alpha(a)$. We claim this preserves IC for the agent while (weakly) improving the principal's payoff when restricting attention to actions that that lead to acceptance probability at least $\hat{p}$ (in either mechanism). IC is preserved because every type of the agent seeks to either minimize or maximize the acceptance probability, and $\alpha(a) \geq \alpha(a') \implies \bar{\alpha}(a) \geq \bar{\alpha}(a')$. Restricting attention to the relevant actions, the principal's payoff gain from using $\bar{\alpha}$ instead of α is

$$\begin{aligned}\int_{p^*}^1 (1-p)u(p)\mathrm{d}F(p) &= \int_{p^*}^1 (1-p)\mathrm{d}\left[\int_{p^*}^p u(p')\mathrm{d}F(p')\right] \\ &= \left[(1-p)\int_{p^*}^p u(p')\mathrm{d}F(p')\right]\Bigg|_{p^*}^1 - \int_{p^*}^1 \left[\int_{p^*}^p u(p')\mathrm{d}F(p')\right]\mathrm{d}(1-p) \\ &= \int_{p^*}^1 \left[\int_{p^*}^p u(p')\mathrm{d}F(p')\right]\mathrm{d}p = \int_{p^*}^1 \left[U(p) - U(p^*)\right]\mathrm{d}p \geq 0,\end{aligned}$$

where the second equality is from integration by parts, and the inequality is because p^* minimizes U. ‖

Case (2): Analogously, for $p < p^*$, consider a modification of α to $\bar{\alpha}$ such that any action a with $\alpha(a) \geq p$ is assigned $\bar{\alpha}(a) = 1$, and for all other actions $\bar{\alpha}(a) = \alpha(a)$. As in the previous case, this modification preserves IC. Taking $p \nearrow p^*$, over the relevant actions the principal's payoff gain from using $\bar{\alpha}$ instead of α is

$$\begin{aligned}\lim_{p\nearrow p^*}\int_p^1 (1-p')u(p')\mathrm{d}F(p') &= \lim_{p\nearrow p^*}\int_p^{p^*} [U(p')-U(p)]\,\mathrm{d}p' + \lim_{p\nearrow p^*}\int_{p^*}^1 [U(p')-U(p)]\,\mathrm{d}p' \\ &= 0 + \int_{p^*}^1 \left[U(p') - \lim_{p\nearrow p^*} U(p)\right] \mathrm{d}p' \\ &= \int_{p^*}^1 \left[U(p') - \inf_{p''} U(p'')\right] \mathrm{d}p' > 0,\end{aligned}$$

where the first equality is analogous to the argument in Case (1), the second equality is by the Dominated Convergence Theorem, and the strict inequality is because $U(p') > \inf_{p''} U(p'')$ for all p' since the infimum is not achieved. In other words, by choosing $p < p^*$ sufficiently close to p^*, the modification strictly improves the principal's payoff (over the relevant actions). ‖

A symmetric argument establishes that a modification of α such that any action a with $\alpha(a) < p^*$ is assigned $\underline{\alpha}(a) = 0$ preserves IC and (weakly) improves the principal's payoff when restricting attention to actions that that lead to acceptance probability less than p^*.

We conclude that the mechanism α^* defined by $\alpha^*(a) = 1$ for all a such that $\alpha(a) \geq \hat{p}$ and $\alpha^*(a) = 0$ for all a such that $\alpha^*(a) < \hat{p}$ preserves IC—thereby inducing the same type-action distribution G—and improves the principal's payoff. Consequently, there is an optimal deterministic mechanism. ■

References

Dessein, W., A. Frankel, and N. Kartik (2023): "Test-Optional Admissions," Unpublished.